

Derivative expansion for the Casimir effect at zero and finite temperature in $d + 1$ dimensions

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(Dated: today)

We apply the derivative expansion approach to the Casimir effect for a real scalar field in d spatial dimensions, to calculate the next to leading order term in that expansion, namely, the first correction to the proximity force approximation. The field satisfies either Dirichlet or Neumann boundary conditions on two static mirrors, one of them flat and the other gently curved. We show that, for Dirichlet boundary conditions, the next to leading order term in the Casimir energy is of quadratic order in derivatives, regardless of the number of dimensions. Therefore it is local, and determined by a single coefficient. We show that the same holds true, if $d \neq 2$, for a field which satisfies Neumann conditions. When $d = 2$, the next to leading order term becomes nonlocal in coordinate space, a manifestation of the existence of a gapless excitation (which do exist also for $d > 2$, but produce sub-leading terms).

We also consider a derivative expansion approach including thermal fluctuations of the scalar field. We show that, for Dirichlet mirrors, the next to leading order term in the free energy is also local for any temperature T . Besides, it interpolates between the proper limits: when $T \rightarrow 0$ it tends to the one we had calculated for the Casimir energy in d dimensions, while for $T \rightarrow \infty$ it corresponds to the one for a theory in $d - 1$ dimensions, because of the expected dimensional reduction at high temperatures. For Neumann mirrors in $d = 3$, we find a nonlocal next to leading order term for any $T > 0$.

PACS numbers: 12.20.Ds, 03.70.+k, 11.10.-z

I. INTRODUCTION

The determination of the Casimir force [1] for a quite general situation, namely, when the geometry of the problem is characterized by two rather arbitrary surfaces, is interesting and potentially useful. Those surfaces may correspond, for example, to the boundaries of two mirrors. Alternatively, the surfaces themselves may describe zero-width (‘thin’) mirrors, which will be the situation considered in this paper. Yet another possibility is that those surfaces may be the interfaces between media with different electromagnetic properties, occupying different spatial regions. In situations like the ones above, it may be convenient to think of the Casimir energy as a *functional* of the functions determining the surfaces. Of course, it is generally quite difficult to compute that functional for arbitrary surfaces. Rather, exact results are only available for highly symmetric configurations, the simplest of which being the case of two flat, infinite, parallel plates. Taking advantage of the simplicity of the result for this highly symmetric configuration, the proximity force approximation (PFA) [2, 3] provides an accurate method to calculate the Casimir energy when the surfaces are gently curved, almost parallel, and close to each other. Introduced by Derjaguin many years ago [2] to compute Van der Waals forces, this approximation consists of replacing both surfaces by a set of parallel plates. The energy is then calculated as the sum of the Casimir energies due to each pair of plates, each plate paired only with the nearest one in the other mirror.

In a recent work [4], we have shown that the PFA can be put into the context of a derivative expansion (DE) for the Casimir energy, when the latter is regarded as a functional of the functions that define the shapes of the mirrors. Indeed, the leading order term in this expansion, which contains no derivatives, does reproduce the PFA, while the higher order ones account for the corrections. In that article we considered, for the sake of simplicity, a massless quantum scalar field satisfying Dirichlet boundary conditions on two surfaces. One of them was assumed to be flat, and such that if coordinates were chosen so that $x_3 = 0$, the other surface could be described by a single function: $x_3 = \psi(x_1, x_2)$.

Since the form of the possible terms in the DE may be determined by dimensional analysis plus symmetry considerations, what is left is the calculation of their respective coefficients. Note that those coefficients are ‘universal’, in the sense that they are independent of the shapes of the mirrors (at least for smooth surfaces). Therefore one can fix those coefficients completely, from the knowledge of their values for a particular surface, or for a family of surfaces. We used, to that effect, a particular family of surfaces, namely, those obtained by an expansion up to the second order in η , which is the (assumed small) departure from flat parallel mirrors: $\psi = a + \eta$, $\eta \ll a$. The coefficients determined from this family were then used to fix the coefficients of the first two terms in the DE, which can then be used to calculate the Casimir energy for more general (but smooth) surfaces.

This approach has been generalized by Bimonte et al [5, 6] in many directions. For example, to the case of two curved, perfectly conducting surfaces, for scalar fields satisfying Dirichlet or Neumann boundary conditions, and also to the electromagnetic case and imperfect boundary conditions. As a validity check, it has been shown that, whenever analytic results are available for particular geometries, the corresponding DE does reproduce both the PFA and its next to leading order (NTLO) correction [4, 5]. The DE approach has also been applied successfully to compute the electrostatic interaction between perfect conductors [7].

In [8], we have extended our previous work [4] to the case of the electromagnetic field coupled to two thin, imperfect mirrors, described by means of the vacuum polarization tensors localized on the mirrors. We have also calculated the NTLO to the PFA static Casimir force. For the particular case of mirrors described by a single dimensionless quantity, we have computed the leading and NTLO corrections as a function of that quantity. We found that the absolute value of the NTLO correction falls down rather quickly for imperfect mirrors [8].

In this article, we apply the DE approach, to models where the fluctuating field is coupled to two perfect mirrors, L and R , at zero or finite temperature, in $d + 1$ spacetime dimensions. Special consideration shall be given to the limiting cases where the fluctuations are either thermal or quantum mechanical, i.e, infinite or zero temperature.

The role of the fluctuating field is played by a massless real scalar field φ , with either Dirichlet or Neumann boundary conditions on both mirrors. This analysis is of interest because of several reasons: on the one hand, as a first step towards incorporating thermal effects into the DE (a fuller treatment should also include finite conductivity corrections along with finite temperature corrections [6]). On the other hand, as we will see, this more general analysis will lead us to a clearer physical picture of the validity of the DE and will also shed some light about possible extensions and improvements.

We will mostly consider the first two terms in the DE; in order to fix their coefficients, we follow the procedure of expanding the vacuum energy up to the second order in η and then extracting the coefficients from the corresponding momentum space kernel. Thus, in what may be considered as a byproduct of our approach, we also present the general result for that kernel, valid for any d , both for the Dirichlet and Neumann cases.

This kernel for the quadratic term in η (regarded now as a field) may be interpreted as a contribution to its 2-point one-particle irreducible function, due to a one-loop φ contribution, which fluctuates satisfying the proper boundary conditions. The coefficients of the DE up to the NTLO result from that kernel, from its expansion up to the second

order in k , the η -field momentum, as one would do in an effective field theory approach [9].

When that expansion to order k^2 does exist, the NTLO terms are of second order in derivatives, and therefore spatially local. However, as in any quantum correction to a 2-point function, we know that non-analytic contributions may arise when the external momentum reaches the threshold to excite modes of the field in the loop. In the present case, since we expand around $k = 0$, those non-analyticities may only originate in the existence of massless modes. When present, they result in contributions which, being nonlocal in space, are similar to the ones that appear in the context of effective field theories, when including the effect of massless virtual particles [9]. We shall see that this kind of non-analyticity does indeed appear, for Neumann conditions, in the form of branch cuts. The physical reason being that when both mirrors impose Neumann conditions there are transverse gapless modes for the fluctuating field. However, except for $d = 2$ and zero temperature ($T = 0$), or $d = 3$ and $T > 0$, those non-analyticities are of higher order than the NTLO, when $k \rightarrow 0$. In this, the only ‘pathological’ case ($d = 2$ and $T = 0$ or $d = 3$ and $T > 0$), the real time version of the kernel has, if rotated to real time, a logarithmic branch cut at zero momentum of the form $k^2 \log(k^2)$, which overcomes the k^2 term (which is also present).

We also study the NTLO term as a function of temperature. In the particular case of $d = 3$, we shall see that for Dirichlet boundary conditions it depends smoothly on the temperature. For Neumann boundary conditions, as an indirect consequence of the non-analyticity at $d = 2$ and zero temperature, the Neumann NTLO term is also non-analytic at any non zero temperature.

This paper is organized as follows. In the next Section we introduce the system and summarize the approach we follow to calculate the free energy Γ_β . In Section III we discuss the DE for Dirichlet boundary conditions on thin, perfect mirrors in $d + 1$ dimensions, discussing the zero and high (infinite) temperature limits. We apply those results to evaluate the Casimir interaction energy between a sphere and a plane at very high temperature. Section IV is devoted to study the DE at zero and high temperatures limits, for a real scalar field with Neumann boundary conditions on the mirrors. The special cases of $d = 2$ with $T = 0$, and $d = 3$ with $T > 0$, are singled out and dealt with in subsection IV B. Higher order terms in the DE are analyzed in Section V. Finally, in Section VI, we summarize our conclusions. The Appendices contain some details of the calculations.

II. THE SYSTEM

We shall adopt Euclidean conventions, whereby the spacetime metric is the identity matrix, and spacetime coordinates are denoted by $x^\mu = x_\mu$ ($\mu = 0, 1, \dots, d$), x_0 being the imaginary time and x_i , ($i = 1, \dots, d$) the spatial Cartesian coordinates.

Regarding the geometry of the system, we shall assume that one of the surfaces, L , is a plane, while the other, R , is such that it can be described by a single Monge patch:

$$L) \ x_d = 0 \quad R) \ x_d = \psi(x_0, x_1, \dots, x_{d-1}) . \quad (1)$$

We have included for R in (1) a more general, time-dependent boundary, in spite of the fact that we are interested in the *static* Casimir effect (SCE). We shall, indeed, at the end of the calculations, impose the condition that the boundaries are time-independent: $\psi = \psi(x_1, \dots, x_{d-1})$, but it turns out to be convenient to keep the more general kind of boundary condition at intermediate stages of the calculation. In this way, the treatment becomes more symmetric, and one may take advantage of that to simplify the calculation. Besides, although it is not our object in this paper, one could rotate back some of the results thus obtained for a non-static ψ to real time, in order to consider a dynamical Casimir effect (DCE) situation.

We follow a functional approach to calculate the free energy $\Gamma_\beta(\psi)$, or its zero temperature limit $E_{\text{vac}}(\psi)$, the vacuum energy. Both are functionals of ψ , that defines the shape of the R mirror (the plane mirror L is assumed to be fixed at $x_d = 0$). Γ_β is also a function of the inverse temperature $\beta \equiv T^{-1}$ (we use units such that Boltzmann constant $k_B = 1$).

In the functional approach, which we shall follow, both objects are obtained by performing a functional integration; indeed:

$$\Gamma_\beta(\psi) = -\frac{1}{\beta} \log \left[\frac{\mathcal{Z}_\beta(\psi)}{\mathcal{Z}_\beta^{(0)}} \right] . \quad (2)$$

where $\mathcal{Z}_\beta(\psi)$ is the partition function; it may be obtained by integrating over field configurations that satisfy the corresponding boundary conditions at L and R , and are also periodic (with period β) in the imaginary time coordinate x_0 (Matsubara formalism). $\mathcal{Z}_\beta^{(0)}$ denotes the partition function in the absence of the mirrors; therefore it corresponds to a relativistic free Bose gas.

$E_{\text{vac}}(\psi)$ is then obtained by taking the limit:

$$E_{\text{vac}}(\psi) = \lim_{\beta \rightarrow \infty} \Gamma_{\beta}(\psi) \equiv \Gamma_{\infty}(\psi) . \quad (3)$$

It is worth noting that in (2) ψ has to be time-independent for $\Gamma_{\beta}(\psi)$ to be a free energy. We do however calculate objects like $\mathcal{Z}_{\beta}(\psi)$ for configurations that may have a time dependence, keeping the same notation.

To avoid an unnecessary repetition of rather similar expressions, we shall write most of the derivations within the context of a finite temperature system, presenting their zero temperature counterparts at the end of the calculations.

We shall consider a real scalar field, with either Dirichlet or Neumann boundary conditions. In both cases, the general setup has a similar structure, but there are also some important differences. Mostly, they come from the different infrared behaviour of their respective Green's functions, and the impact that that behaviour has on the DE. Accordingly, we present them in two separate sections.

III. DIRICHLET BOUNDARY CONDITIONS

We start from the functional representation of $\mathcal{Z}_{\beta}(\psi)$:

$$\mathcal{Z}_{\beta}(\psi) = \int \mathcal{D}\varphi \, \delta_L(\varphi) \, \delta_R(\varphi) \, e^{-\mathcal{S}_0(\varphi)} , \quad (4)$$

where $\delta_A(\varphi)$, $A = L, R$, is a functional δ function which imposes Dirichlet boundary conditions on the respective mirror, while \mathcal{S}_0 is the free Euclidean action for a massless real scalar field in $d+1$ dimensions, at finite temperature:

$$\mathcal{S}_0 = \frac{1}{2} \int_0^{\beta} dx_0 \int d^d x \, (\partial\varphi)^2 , \quad (5)$$

with periodic conditions for φ in the time-like coordinate, namely, $\varphi(x_0, \mathbf{x}) = \varphi(x_0 + \beta, \mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^{(d)}$.

To proceed, one should then exponentiate the δ -functionals by introducing two auxiliary fields, λ_L and λ_R , functions of $x_{\parallel} \equiv (x_0, x_1, \dots, x_{d-1}) \equiv (x_0, \mathbf{x}_{\parallel})$, also satisfying periodic boundary conditions in the x_0 coordinate. In the Dirichlet case, we have:

$$\begin{aligned} \delta_L(\varphi) &= \int \mathcal{D}\lambda_L \, e^{i \int d^d x_{\parallel} \, \lambda_L(x_{\parallel}) \varphi(x_{\parallel}, 0)} \\ \delta_R(\varphi) &= \int \mathcal{D}\lambda_R \, e^{i \int d^d x_{\parallel} \, \sqrt{g(x_{\parallel})} \, \lambda_R(x_{\parallel}) \varphi(x_{\parallel}, \psi(x_{\parallel}))} , \end{aligned} \quad (6)$$

where g is the determinant of $g_{\alpha\beta}$, the induced metric on R :

$$g_{\alpha\beta}(x_{\parallel}) = \delta_{\alpha\beta} + \partial_{\alpha}\psi(x_{\parallel})\partial_{\beta}\psi(x_{\parallel}) , \quad (7)$$

$$\Rightarrow g(x_{\parallel}) = 1 + (\partial\psi(x_{\parallel}))^2 . \quad (8)$$

We have adopted the convention that indices from the beginning of the Greek alphabet run from 0 to $d-1$.

Using the exponential representations above in (4), one derives the alternative expression:

$$\mathcal{Z}_{\beta}(\psi) = \int \mathcal{D}\varphi \, \mathcal{D}\lambda_L \mathcal{D}\lambda_R \, e^{-\mathcal{S}_0(\varphi) + i \int d^{d+1}x \, J_D(x) \varphi(x)} , \quad (9)$$

where the ‘Dirichlet current’ $J_D(x)$ is given by:

$$J_D(x) = \lambda_L(x_{\parallel}) \delta(x_3) + \lambda_R(x_{\parallel}) \sqrt{g(x_{\parallel})} \delta(x_3 - \psi(x_{\parallel})) . \quad (10)$$

It is possible to get rid of the $\sqrt{g(x_{\parallel})}$ factor above just by redefining λ_R : $\lambda_R(x_{\parallel}) \rightarrow \lambda_R(x_{\parallel})/\sqrt{g(x_{\parallel})}$. This redefinition induces a nontrivial Jacobian. However, this Jacobian is independent of the distance between the mirrors, therefore irrelevant to the calculation of their relative Casimir force; hence we discard it.

The integral over φ , a Gaussian, yields:

$$\mathcal{Z}_{\beta}(\psi) = \mathcal{Z}_{\beta}^{(0)} \int \mathcal{D}\lambda_L \mathcal{D}\lambda_R \, e^{-\frac{1}{2} \int_{x_{\parallel}, x'_{\parallel}} \lambda_A(x_{\parallel}) \mathbb{T}_{AB}(x_{\parallel}, x'_{\parallel}) \lambda_B(x'_{\parallel})} , \quad (11)$$

where we have introduced the objects:

$$\mathbb{T}_{LL}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | x'_{\parallel}, 0 \rangle \quad (12)$$

$$\mathbb{T}_{LR}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | x'_{\parallel}, \psi(x'_{\parallel}) \rangle \quad (13)$$

$$\mathbb{T}_{RL}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^2)^{-1} | x'_{\parallel}, 0 \rangle \quad (14)$$

$$\mathbb{T}_{RR}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^2)^{-1} | x'_{\parallel}, \psi(x'_{\parallel}) \rangle \quad (15)$$

where we use a “bra-ket” notation to denote matrix elements of operators, and

$$\langle x | (-\partial^2)^{-1} | y \rangle = \sum_{n=-\infty}^{\infty} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i(\omega_n(x_0-y_0) + \mathbf{k} \cdot (\mathbf{x}-\mathbf{y}))}}{(\omega_n^2 + \mathbf{k}^2)} \equiv \Delta(x-y), \quad (16)$$

where we have introduced the Matsubara frequencies: $\omega_n \equiv \frac{2\pi n}{\beta}$, $n \in \mathbb{Z}$. The free energy $\Gamma_{\beta}(\psi)$ is then

$$\Gamma_{\beta}(\psi) = \frac{1}{2\beta} \text{Tr} \log \mathbb{T}. \quad (17)$$

where ψ is regarded as time independent, something that one can impose at the end of the calculation.

Γ_{β} still contains ‘self-energy’ contributions, i.e., contributions invariant under the rigid displacement $\psi(x_{\parallel}) \rightarrow \psi(x_{\parallel}) + \epsilon$. Since we are just interested in the Casimir force we shall neglect them altogether whenever they emerge in the calculations below.

A. Derivative expansion

The DE is implemented by following the same idea and approach introduced in [4]. The calculation is, in many aspects, identical to the one in [4], with the only differences in the number of dimensions and in the fact that the time coordinate is compact (periodic), so frequency integrations have to be replaced by sum over Matsubara frequencies. Therefore, we do not repeat all the steps presented there, rather, we limit ourselves to convey the relevant results.

First, we note that in the DE approach applied to this case, keeping up to two derivatives, the Casimir free energy can be written as follows:

$$\Gamma_{\beta}(\psi) = \int d^{d-1} \mathbf{x}_{\parallel} \left\{ b_0\left(\frac{\psi}{\beta}\right) \frac{1}{[\psi(\mathbf{x}_{\parallel})]^d} + b_2\left(\frac{\psi}{\beta}\right) \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_{\parallel})]^d} \right\} \quad (18)$$

where the two dimensionless functions b_0 and b_2 can be obtained from the knowledge of the Casimir free energy for small departures around the $\psi(\mathbf{x}_{\parallel}) = a = \text{constant}$ case. Indeed, setting:

$$\psi(\mathbf{x}_{\parallel}) = a + \eta(\mathbf{x}_{\parallel}), \quad (19)$$

one expands Γ_{β} in (17) in powers of η , up to the second order. Thus,

$$\Gamma_{\beta}(a, \eta) = \Gamma_{\beta}^{(0)}(a) + \Gamma_{\beta}^{(1)}(a, \eta) + \Gamma_{\beta}^{(2)}(a, \eta) + \dots \quad (20)$$

where the index denotes the order in η .

For the expansion above, $\Gamma_{\beta}^{(0)}$ is proportional to the area of the mirrors, L^{d-1} . In terms of the Matsubara frequencies: $\omega_n \equiv \frac{2\pi n}{\beta}$, $n \in \mathbb{Z}$, the explicit form of the zero order term per unit area is as follows:

$$\begin{aligned} \frac{\Gamma_{\beta}^{(0)}(a)}{L^{d-1}} &= \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}_{\parallel}}{(2\pi)^{d-1}} \log \left[1 - e^{-2a\sqrt{\omega_n^2 + \mathbf{p}_{\parallel}^2}} \right] \\ &= \frac{1}{a^d} b_0\left(\frac{a}{\beta}\right), \end{aligned} \quad (21)$$

where:

$$\begin{aligned} b_0(\xi) &= \frac{\xi}{2} \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}_{\parallel}}{(2\pi)^{d-1}} \log \left[1 - e^{-2\sqrt{(2\pi n \xi)^2 + \mathbf{p}_{\parallel}^2}} \right] \\ &= \xi \frac{\pi^{(1-d)/2}}{2^{d-1} \Gamma(\frac{d-1}{2})} \sum_{n=-\infty}^{+\infty} \int_0^{\infty} d\rho \rho^{d-2} \log \left[1 - e^{-2\sqrt{(2\pi n \xi)^2 + \rho^2}} \right] \end{aligned} \quad (22)$$

is the dimensionless function which appears in the DE for the zero order term ($\xi = a/\beta$).

Regarding $\Gamma_\beta^{(2)}$, which is necessary in order to find b_2 , the result can be presented in a more compact form in terms of its Fourier space version. Defining the spatial Fourier transform of η by:

$$\eta(\mathbf{x}_\parallel) = \int \frac{d^{d-1}\mathbf{k}_\parallel}{(2\pi)^{d-1}} e^{i\mathbf{k}_\parallel \cdot \mathbf{x}_\parallel} \tilde{\eta}(\mathbf{k}_\parallel) \quad (23)$$

we have

$$\Gamma_\beta^{(2)} = \frac{1}{2} \int \frac{d^{d-1}\mathbf{k}_\parallel}{(2\pi)^{d-1}} f^{(2)}(0, \mathbf{k}_\parallel) |\tilde{\eta}(\mathbf{k}_\parallel)|^2 \quad (24)$$

where

$$\begin{aligned} f^{(2)}(\omega_n, \mathbf{k}_\parallel) &= -\frac{2}{\beta} \sum_{m=-\infty}^{+\infty} \int \frac{d^{d-1}\mathbf{p}_\parallel}{(2\pi)^{d-1}} \sqrt{\omega_m^2 + \mathbf{p}_\parallel^2} \sqrt{(\omega_m + \omega_n)^2 + (\mathbf{p}_\parallel + \mathbf{k}_\parallel)^2} \\ &\times \frac{1}{1 - \exp(-2a\sqrt{\omega_m^2 + \mathbf{p}_\parallel^2})} \frac{1}{\exp[2a\sqrt{(\omega_m + \omega_n)^2 + (\mathbf{p}_\parallel + \mathbf{k}_\parallel)^2}] - 1} \\ &\equiv a^{-(d+2)} F^{(2)}\left(\frac{a}{\beta}; n, a|\mathbf{k}_\parallel|\right) \end{aligned} \quad (25)$$

with

$$\begin{aligned} F^{(2)}(\xi; n, |\mathbf{l}_\parallel|) &= -2\xi \sum_{m=-\infty}^{+\infty} \int \frac{d^{d-1}\mathbf{p}_\parallel}{(2\pi)^{d-1}} \left\{ \sqrt{(2\pi m\xi)^2 + \mathbf{p}_\parallel^2} \sqrt{(2\pi(m+n)\xi)^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2} \right. \\ &\times \left. \frac{1}{1 - \exp[-2\sqrt{(2\pi m\xi)^2 + \mathbf{p}_\parallel^2}]} \frac{1}{\exp\{2\sqrt{[2\pi(m+n)\xi]^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2}\} - 1} \right\}, \end{aligned} \quad (26)$$

which is also a dimensionless function with dimensionless arguments. We have made explicit the fact that the result will only depend on the modulus of \mathbf{l}_\parallel , as any dependence on its direction may be got rid off by a redefinition of the integration variables.

The coefficient b_2 can be immediately defined in terms of $F^{(2)}$. Indeed,

$$b_2(\xi) = \frac{1}{2} \left[\frac{\partial F^{(2)}(\xi; n, |\mathbf{l}_\parallel|)}{\partial |\mathbf{l}_\parallel|^2} \right]_{n \rightarrow 0, |\mathbf{l}_\parallel| \rightarrow 0}. \quad (27)$$

In the following subsection, we consider the low and high temperature limits of the two coefficients b_0 and b_2 ; since they determine the form of the DE in the corresponding limits. We note that the relevant scale to compare the temperature with is the inverse of the distance between the mirrors. Thus, in terms of the variable ξ , the relevant cases are: $\xi \rightarrow 0$ (zero temperature limit) and $\xi \rightarrow \infty$ (infinite temperature limit). We discuss them below.

1. The zero and high temperature limits

The zero temperature limit corresponds to $\xi \rightarrow 0$, and it can be implemented by replacing a sum over discrete indices by an integral over a continuous index. Defining $k_0 = 2\pi n\xi$, we get an integral over k_0 ; the Jacobian being $1/(2\pi\xi)$. The results for the two coefficients, in d dimensions (we introduce the number of dimensions as an explicit parameter), are:

$$[b_0(d, \xi)]_{\xi \ll 1} \sim \frac{1}{2} \int \frac{d^d p_\parallel}{(2\pi)^d} \log[1 - e^{-2|p_\parallel|}] \equiv b_0(d), \quad (28)$$

and

$$[b_2(d, \xi)]_{\xi \ll 1} = \frac{1}{2} \left[\frac{\partial F_0^{(2)}(d, |l_\parallel|)}{\partial |l_\parallel|^2} \right]_{l_\parallel \rightarrow 0} \equiv b_2(d), \quad (29)$$

	$\frac{b_2(d)}{b_0(d)}$	\sim
$d = 1$	$\frac{1}{\pi^2} (1 + \frac{\pi^2}{3})$	0.435
$d = 2$	$\frac{1+6\zeta(3)}{12\zeta(3)}$	0.569
$d = 3$	$2/3$	0.667
$d = 4$	$\frac{-\zeta(3)+10\zeta(5)}{12\zeta(5)}$	0.737
$d = 5$	$\frac{10\pi^2-21}{10\pi^2}$	0.787
$d = 6$	$\frac{-2\zeta(5)+7\zeta(7)}{6\zeta(7)}$	0.824

TABLE I. Values of the ratios $\frac{b_2(d)}{b_0(d)}$ for the lowest dimensions.

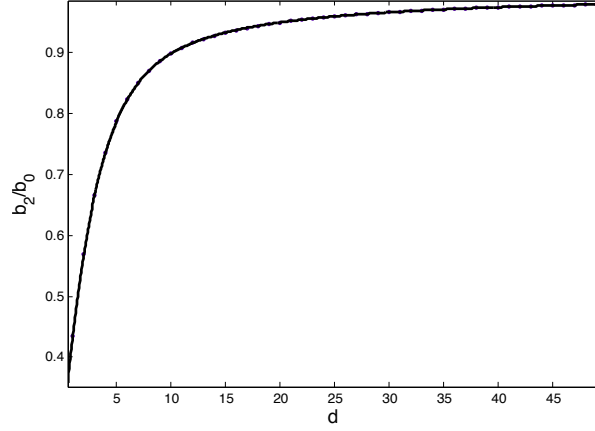


FIG. 1. The function b_2/b_0 tends to 1 for large values of d .

where:

$$F_0^{(2)}(d, |l_{\parallel}|) = -2 \int \frac{d^d p_{\parallel}}{(2\pi)^d} |p_{\parallel}| |p_{\parallel} + l_{\parallel}| \frac{1}{1 - e^{-2|p_{\parallel}|}} \frac{1}{e^{2|p_{\parallel}+l_{\parallel}|} - 1} . \quad (30)$$

It is possible to give a more explicit expression for $b_2(d)$, since this coefficient may be obtained by taking derivatives inside of the integrand of (30). Also, the $b_0(d)$ coefficient can be exactly evaluated as a function of d . We present, in the following table, the ratio between the two coefficients as a function of d , for $1 \leq d \leq 6$:

It is interesting to remark that the relative weight of the NTLO correction grows with the number of dimensions. Indeed, the general results for $b_0(d)$ and $b_2(d)$ in an arbitrary number of dimensions are:

$$b_0(d) = - \frac{\Gamma((d+1)/2) \zeta(d+1)}{(4\pi)^{(d+1)/2}} \quad (31)$$

and

$$b_2(d) = - \frac{1}{12\pi^2} \frac{\pi^{d/2}}{2^d} \left[\frac{(d-3)(d-1)}{d} \Gamma\left(2 - \frac{d}{2}\right) \zeta(2-d) + \pi^{3/2-d} (d+1) \Gamma\left(\frac{d+1}{2}\right) \zeta(d+1) \right] \quad (32)$$

These expressions are consistent with those derived in Ref.[10], using a different method and in the context of the dynamical Casimir effect. Figure 1 shows that the ratio $b_2(d)/b_0(d)$ is an increasing function of d , tending to 1 as $d \rightarrow \infty$.

Let us now consider the very high (infinite) temperature limit. When $\xi \gg 1$, we see that only the $n = 0$ term in the sum representing b_0 yields a non-vanishing contribution,

$$b_0(d, \xi) \sim \frac{\xi}{2} \int \frac{d^{d-1} \mathbf{p}_{\parallel}}{(2\pi)^{d-1}} \log(1 - e^{-2|\mathbf{p}_{\parallel}|}) , \quad (33)$$

or, introducing explicitly the dependence on the number of space dimensions, d ,

$$[b_0(\xi, d)]_{\xi \gg 1} \sim \xi [b_0(\xi, d-1)]_{\xi \rightarrow 0} = b_0(d-1), \quad (34)$$

a reflection of the well known ‘dimensional reduction’ phenomenon at high temperatures, for bosonic degrees of freedom.

For the b_2 coefficient, a similar analysis shows that only $m = 0$ has to be kept, and:

$$[b_2(\xi, d)]_{\xi \gg 1} \sim \xi [b_2(\xi, d-1)]_{\xi \rightarrow 0} = b_2(d-1). \quad (35)$$

Putting together (34) and (35), we finally get for the DE up to the second order in the high temperature limit:

$$[\Gamma_\beta(\psi, d)]_{\psi/\beta \gg 1} \sim \frac{1}{\beta} \int d^{d-1} \mathbf{x}_\parallel \left\{ b_0(d-1) \frac{1}{[\psi(\mathbf{x}_\parallel)]^{d-1}} + b_2(d-1) \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_\parallel)]^{d-1}} \right\}. \quad (36)$$

In particular, the free energy reads, for $d = 3$,

$$[\Gamma_\beta(\psi, 3)]_{\psi/\beta \gg 1} \sim -\frac{\zeta(3)}{16\pi\beta} \int d^2 \mathbf{x}_\parallel \frac{1}{[\psi(\mathbf{x}_\parallel)]^2} \{1 + 0.569 (\partial\psi)^2\}. \quad (37)$$

As an example, let us now apply the results above to the evaluation of the Casimir interaction between a sphere and a plane at very high temperatures. The sphere has radius R , and is in front of a plane at a minimum distance a ($a \ll R$). Although the surface of the sphere cannot be covered by a single function $z = \psi(x_\parallel)$, as in previous works, we will nevertheless consider just the region of the sphere which is closer to the plane [4]. We shall see that this procedure still produces results which are quantitatively adequate within the present approximation and assumptions, even beyond the lowest order.

The function ψ is

$$\psi(\rho) = a + R \left(1 - \sqrt{1 - \frac{\rho^2}{R^2}} \right), \quad (38)$$

where we are using polar coordinates (ρ, ϕ) for the $x_3 = 0$ plane. This function describes the hemisphere when $0 \leq \rho \leq R$. The DE will be well defined if we restrict the integrations to the region $0 \leq \rho \leq \rho_M < R$. We will assume that $\rho_M/R = O(1) < 1$. Inserting this expression for ψ into Eq.(37) and performing explicitly the integrations we obtain

$$[\Gamma_\beta(\psi, 3)]_{\psi/\beta \gg 1} \sim -\frac{\zeta(3)R}{8\beta a} \left(1 + 0.569 \frac{a}{R} \log \left(\frac{a}{R} \right) \right). \quad (39)$$

Note that, as long as $a \ll R$, the force will not depend on ρ_M . As expected on dimensional grounds, the R/a^2 behavior of the leading contribution in the zero temperature case changes to $R/a\beta$ at very high temperatures. The NTLO correction is analytic when written in terms of derivatives of the function ψ , but non-analytic in $\frac{a}{R}$. This behavior has been already noted in numerical estimations of the Casimir interaction between a sphere and a plane in the infinite temperature limit, for the electromagnetic case in Ref.[11].

It is interesting to remark that the expression for the free energy at high temperatures in $d = 3$ is quite similar to that corresponding to the electrostatic force F_z between two surfaces held at a constant potential difference V [7]

$$F_z \simeq -\frac{\epsilon_0 V^2}{2} \int d^2 x_\parallel \frac{1}{\psi^2} \left[1 + \frac{1}{3} (\partial\psi)^2 \right]. \quad (40)$$

Therefore, when considering an arbitrary surface over a plane, the high temperature limit of the free energy will have the same behavior than the electrostatic force. For instance, from the results of [7], for a cylinder of radius R and length L at a distance a of a plane, we expect the leading term of the free energy to be proportional to $\frac{L}{\beta} \sqrt{\frac{R}{a^3}}$ while its NTLO correction must be a coefficient times $\frac{L}{\beta} \sqrt{\frac{1}{aR}}$.

Going back to the general case, at intermediate temperatures the coefficients $b_2(\xi)$ given by Eq.(27) and $b_0(\xi)$ of Eq.(22) should interpolate between their zero and high temperatures values. This is shown in Figs. 2 and 3, where we plot the ratio b_2/b_0 as a function of the dimensionless temperature ξ for $d = 3$ and $d = 4$, respectively. The plot for $d = 3$ interpolates between the value $b_2/b_0 = 0.67$ for zero temperature, and 0.57 at high temperatures. On the other hand, for $d = 4$, b_2/b_0 interpolates between 0.74 for zero temperature and 0.67 at high temperatures. These limits are

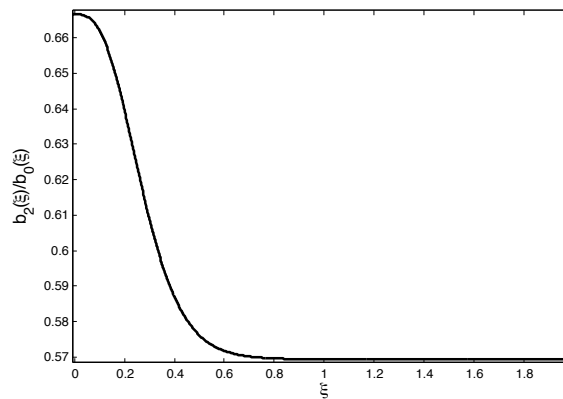


FIG. 2. Ratio between coefficient $b_2(\xi)$ given by Eq.(27) and the coefficient $b_0(\xi)$ of Eq.(22), as a function of the dimensionless temperature ξ for $d = 3$. The plot interpolates between the value $b_2/b_0 = 0.67$ for zero temperature, and 0.57 at high temperatures.

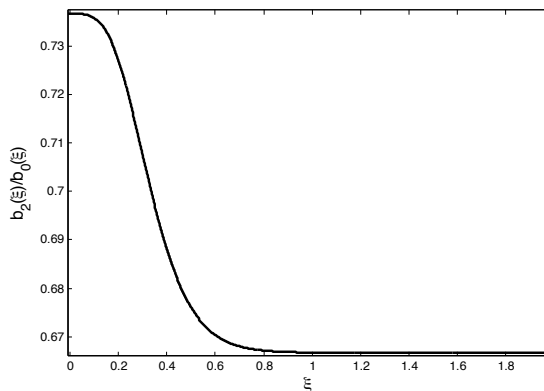
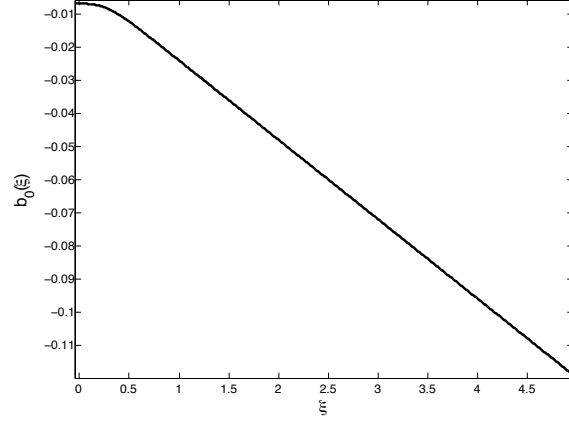
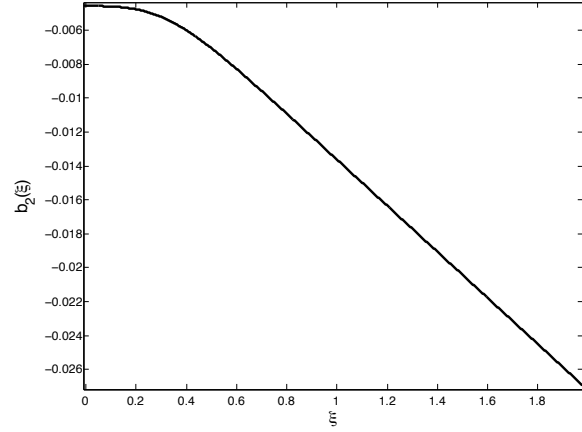


FIG. 3. Ratio between coefficient $b_2(\xi)$ given by Eq.(27) and the coefficient $b_0(\xi)$ of Eq.(22), as a function of the dimensionless temperature ξ for $d = 4$. The plot interpolates between the value $b_2/b_0 = 0.74$ for zero temperature, and 0.67 at high temperatures.

in agreement with the results in Table 1. The ratio b_2/b_0 gives a quantitative measure of the relevance of the NTLO correction to the PFA. Note that, both for $d = 3$ and $d = 4$, it converges quickly to the infinite temperature value.

Finally, it is worth stressing that, as the coefficients b_0 and b_2 are functions of $\xi = T\psi$, the evaluation of these functions is crucial in order to compute the Casimir free energy using the DE in any concrete example, at a fixed temperature. The previous plots describe the dependence of their ratio with distance, at a fixed temperature. In Fig. 4 we plot $b_0(\xi)$ in $3 + 1$ dimensions. We can see that, at low temperatures, the curve is very flat. Indeed, it is well known that the low temperature corrections to the free energy for parallel plates are proportional to ξ^3 for $\xi \ll 1$, and this behavior is well reproduced in the numerical evaluation. Moreover, the function $b_0(\xi)$ acquires very quickly the linear behavior expected at very high temperatures. For the sake of completeness, in Fig. 5, we plot the function $b_2(\xi)$, which has similar characteristics.

These results may be useful to understand the nontrivial interplay between geometry and temperature for open geometries, like the sphere-plate and the cylinder-plate configurations, described in Ref.[12]. Indeed, it was pointed out there that local approximation techniques such as the PFA are generically inapplicable at low temperatures. From our results we see that both functions b_0 and b_2 approach their high temperature behavior for relatively low values of ξ . Therefore it would not be valid to insert the low- ξ expansions of these functions into Eq.(18), and then apply the result to open geometries for which the condition $\xi \ll 1$ is not satisfied.

FIG. 4. The function $b_0(\xi)$ in $3 + 1$ dimensions.FIG. 5. The function $b_2(\xi)$ in $3 + 1$ dimensions.

IV. NEUMANN BOUNDARY CONDITIONS

Again we start from the functional representation of $\mathcal{Z}_\beta(\psi)$ given in (4), but now we use the functional δ functions which impose Neumann (rather than Dirichlet) boundary conditions on the two mirrors. We assume the mirrors to be characterized by the same surfaces we used in the Dirichlet case.

The boundary conditions may be written as follows:

$$\begin{aligned} L) [\partial_d \varphi(x_\parallel, x_d)]_{x_d=0} &= 0 \\ R) [\partial_n \varphi(x_\parallel, x_d)]_{x_d=\psi(x_\parallel)} &= 0, \end{aligned} \quad (41)$$

where $\partial_n = n^\mu \partial_\mu$, with n^μ the unit normal to the surface $x_d - \psi(x_\parallel) = 0$:

$$\begin{aligned} n^\mu(x_\parallel) &= \frac{N^\mu(x_\parallel)}{|N(x_\parallel)|} \\ N^\mu(x_\parallel) &= \delta_d^\mu - \delta_\alpha^\mu \partial_\alpha \psi(x_\parallel), \end{aligned} \quad (42)$$

and $|N(x_\parallel)| = \sqrt{g(x_\parallel)}$.

To exponentiate the δ -functionals, we again introduce two auxiliary fields, λ_L and λ_R :

$$\begin{aligned}\delta_L(\varphi) &= \int \mathcal{D}\lambda_L e^{i \int d^d x_{\parallel} \lambda_L(x_{\parallel}) [\partial_d \varphi(x_{\parallel}, x_d)]_{x_d=0}} \\ \delta_R(\varphi) &= \int \mathcal{D}\lambda_R e^{i \int d^d x_{\parallel} \sqrt{g(x_{\parallel})} \lambda_R(x_{\parallel}) [\partial_n \varphi(x_{\parallel}, x_d)]_{x_d=\psi(x_{\parallel})}} \\ &= \int \mathcal{D}\lambda_R e^{i \int d^d x_{\parallel} \lambda_R(x_{\parallel}) [\partial_N \varphi(x_{\parallel}, x_d)]_{x_d=\psi(x_{\parallel})}},\end{aligned}\quad (43)$$

where we introduced the notation $\partial_N = N^\mu \partial_\mu$. Thus, using those exponential representations, we derive:

$$\mathcal{Z}_\beta(\psi) = \int \mathcal{D}\varphi \mathcal{D}\lambda_L \mathcal{D}\lambda_R e^{-S_0(\varphi) + i \int d^{d+1}x J_N(x) \varphi(x)}, \quad (44)$$

where, by analogy with the Dirichlet case, we introduce the ‘current’ $J_N(x)$:

$$J_N(x) = \lambda_L(x_{\parallel}) \partial_d \delta(x_d) + \lambda_R(x_{\parallel}) \partial_N \delta(x_d - \psi(x_{\parallel})). \quad (45)$$

Note that there is no need to get rid now of any metric-dependent factor, as we did for the Dirichlet case.

The integral over φ becomes then:

$$\mathcal{Z}_\beta(\psi) = \mathcal{Z}_\beta^{(0)} \int \mathcal{D}\lambda_L \mathcal{D}\lambda_R e^{-\frac{1}{2} \int_{x_{\parallel}, x'_{\parallel}} \lambda_A(x_{\parallel}) \mathbb{U}_{AB}(x_{\parallel}, x'_{\parallel}) \lambda_B(x'_{\parallel})}, \quad (46)$$

where:

$$\begin{aligned}\mathbb{U}_{LL}(x_{\parallel}, x'_{\parallel}) &= - \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{i k_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} \frac{|k_{\parallel}|}{2} \\ \mathbb{U}_{LR}(x_{\parallel}, x'_{\parallel}) &= - \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{i k_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} e^{-|k_{\parallel}| \psi(x'_{\parallel})} \frac{|k_{\parallel}| + i k_{\parallel} \cdot \partial \psi(x'_{\parallel})}{2} \\ \mathbb{U}_{RL}(x_{\parallel}, x'_{\parallel}) &= - \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{i k_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} e^{-|k_{\parallel}| \psi(x_{\parallel})} \frac{|k_{\parallel}| - i k_{\parallel} \cdot \partial \psi(x_{\parallel})}{2} \\ \mathbb{U}_{RR}(x_{\parallel}, x'_{\parallel}) &= \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{i k_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} e^{-|k_{\parallel}| [\psi(x_{\parallel}) - \psi(x'_{\parallel})]} \\ &\quad \times \frac{1}{2} \left\{ -|k_{\parallel}| - i k_{\parallel} \cdot [\partial \psi(x_{\parallel}) + \partial \psi(x'_{\parallel})] + \frac{1}{|k_{\parallel}|} (k_{\parallel} \cdot \partial \psi(x_{\parallel}) k_{\parallel} \cdot \partial \psi(x'_{\parallel})) \right\}\end{aligned}\quad (47)$$

The free energy $\Gamma_\beta(\psi)$ is then

$$\Gamma_\beta(\psi) = \frac{1}{2\beta} \text{Tr} \log \mathbb{U}, \quad (48)$$

which, as in the Dirichlet case, does contain ‘self-energy’ contributions, to be discarded here by the same reason as there. Again, Ψ is assumed to be time independent.

A. Derivative expansion

Assuming that one could proceed as in the Dirichlet case, keeping up to two derivatives, the derivative expanded Casimir free energy could be written as follows:

$$\Gamma_\beta(\psi) = \int d^{d-1} \mathbf{x}_{\parallel} \left\{ c_0 \left(\frac{\psi}{\beta} \right) \frac{1}{[\psi(\mathbf{x}_{\parallel})]^d} + c_2 \left(\frac{\psi}{\beta} \right) \frac{(\partial \psi)^2}{[\psi(\mathbf{x}_{\parallel})]^d} \right\} \quad (49)$$

with two new dimensionless functions c_0 and c_2 . Those coefficients may be determined from the knowledge of the Neumann Casimir free energy for small departures around the $\psi(\mathbf{x}_{\parallel}) = a = \text{constant}$ case, up to the second order in the departure. Again, we focus on the cases of purely quantum or purely thermal effects, except for the more realistic case of $d = 3$. As we will show in what follows, the NTLO term is quadratic, except when $d = 2$ at zero temperature, or when $d = 3$ and there is a non zero (finite or infinite) temperature.

We present the calculation of the terms contributing to that expansion in Appendix A.

It is quite straightforward to see that the zero order term coincides with the one for the Dirichlet case; namely: $c_0 = b_0$.

$\Gamma_\beta^{(2)}$ has the following form:

$$\Gamma_\beta^{(2)} = \frac{1}{2} \int \frac{d^{d-1} \mathbf{k}_\parallel}{(2\pi)^{d-1}} g^{(2)}(0, \mathbf{k}_\parallel) |\tilde{\eta}(\mathbf{k}_\parallel)|^2 \quad (50)$$

where

$$\begin{aligned} g^{(2)}(\omega_n, \mathbf{k}_\parallel) &= -\frac{2}{\beta} \sum_{m=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}_\parallel}{(2\pi)^{d-1}} \frac{[\omega_m(\omega_m + \omega_n) + \mathbf{p}_\parallel \cdot (\mathbf{p}_\parallel + \mathbf{k}_\parallel)]^2}{\sqrt{\omega_m^2 + \mathbf{p}_\parallel^2} \sqrt{(\omega_m + \omega_n)^2 + (\mathbf{p}_\parallel + \mathbf{k}_\parallel)^2}} \\ &\times \frac{1}{1 - \exp(-2a\sqrt{\omega_m^2 + \mathbf{p}_\parallel^2})} \frac{1}{\exp[2a\sqrt{(\omega_m + \omega_n)^2 + (\mathbf{p}_\parallel + \mathbf{k}_\parallel)^2}] - 1} \\ &= a^{-(d+2)} G^{(2)}\left(\frac{a}{\beta}; n, a|\mathbf{k}_\parallel|\right) \end{aligned} \quad (51)$$

with

$$\begin{aligned} G^{(2)}(\xi; n, |\mathbf{l}_\parallel|) &= -2\xi \sum_{m=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}_\parallel}{(2\pi)^{d-1}} \frac{[(2\pi\xi)^2 m(m+n) + \mathbf{p}_\parallel \cdot (\mathbf{p}_\parallel + \mathbf{l}_\parallel)]^2}{\sqrt{(2\pi\xi)^2 m^2 + \mathbf{p}_\parallel^2} \sqrt{(2\pi\xi)^2 (m+n)^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2}} \\ &\times \frac{1}{1 - \exp(-2\sqrt{(2\pi\xi)^2 m^2 + \mathbf{p}_\parallel^2})} \frac{1}{\exp[2\sqrt{(2\pi\xi)^2 (m+n)^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2}] - 1} . \end{aligned} \quad (52)$$

$$c_2(\xi) = \frac{1}{2} \left[\frac{\partial G^{(2)}(\xi; n, |\mathbf{l}_\parallel|)}{\partial |\mathbf{l}_\parallel|^2} \right]_{n \rightarrow 0, |\mathbf{l}_\parallel| \rightarrow 0} . \quad (53)$$

1. The zero and high temperature limits

As before, the zero temperature limit can be implemented by replacing a sum over discrete indices by an integral over a continuous index. The result for the coefficient c_2 in d dimensions is:

$$[c_2(d, \xi)]_{\xi \ll 1} = \left[\frac{\partial G_0^{(2)}(|l_\parallel|)}{\partial |l_\parallel|^2} \right]_{l_\parallel \rightarrow 0} \equiv c_2(d) , \quad (54)$$

where:

$$G_0^{(2)}(d, |l_\parallel|) = -2 \int \frac{d^d \mathbf{p}_\parallel}{(2\pi)^d} \frac{[\mathbf{p}_\parallel \cdot (\mathbf{p}_\parallel + \mathbf{l}_\parallel)]^2}{|\mathbf{p}_\parallel| |\mathbf{p}_\parallel + \mathbf{l}_\parallel|} \frac{1}{1 - e^{-2|\mathbf{p}_\parallel|}} \frac{1}{e^{2|\mathbf{p}_\parallel + \mathbf{l}_\parallel|} - 1} . \quad (55)$$

For $d = 1$, the coefficient c_2 coincides with its Dirichlet counterpart b_2 . In higher dimensions, the structure of the form factor is different. We present, in Table 2, the ratio between $c_2(d)$ and $c_0(d) \equiv b_0(d)$ as a function of d , for $d \neq 2$:

We see that the ratio $c_2(d)/c_0(d)$ is non-monotonous and always negative for $d \neq 1$. We have also checked that $c_2(d)/c_0(d) \rightarrow -1$ for large values of d .

In the particular case $d = 2$ is not possible to compute the coefficient by introducing the derivative with respect to $|l_\parallel|^2$ inside the integral in Eq.(55), because of infrared divergences. This is a signal of a branch cut at zero momentum, as we will show in Section IV B.

The high temperature limit can be obtained, as for the Dirichlet case, taking the limit $\xi \gg 1$. “Dimensional reduction” takes place, and the free energy becomes

$$[\Gamma_\beta(\psi, d)]_{\psi/\beta \gg 1} \sim \frac{1}{\beta} \int d^{d-1} \mathbf{x}_\parallel \left\{ b_0(d-1) \frac{1}{[\psi(\mathbf{x}_\parallel)]^{d-1}} + c_2(d-1) \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_\parallel)]^{d-1}} \right\} . \quad (56)$$

Regarding intermediate temperatures, Fig. 6 shows the ratio between coefficients $c_2(\xi)$ and $c_0(\xi)$, as a function of the dimensionless temperature ξ , for the Neumann boundary condition in $d = 4$ dimensions. The plot interpolates between the value $c_2/c_0 = -1.00$ and -1.36 at zero and high temperatures respectively, in agreement with the results shown in Table 2.

	$\frac{c_2(d)}{c_0(d)}$	\sim
$d = 1$	$\frac{1}{3} - \frac{\zeta(0)}{3\zeta(2)}$	0.435
$d = 3$	$\frac{2}{3} - \frac{4\zeta(2)}{3\zeta(4)}$	-1.360
$d = 4$	$\frac{5}{6} - \frac{19\zeta(3)}{12\zeta(5)}$	-1.002
$d = 5$	$1 - \frac{9\zeta(4)}{5\zeta(6)}$	-0.915
$d = 6$	$\frac{7}{6} - \frac{2\zeta(5)}{\zeta(7)}$	-0.890
$d = 7$	$\frac{4}{3} - \frac{46\zeta(6)}{\zeta(8)}$	-0.886

TABLE II. Values of the ratios $\frac{c_2(d)}{c_0(d)}$ for the lowest dimensions. Note that we have excluded the case $d = 2$.

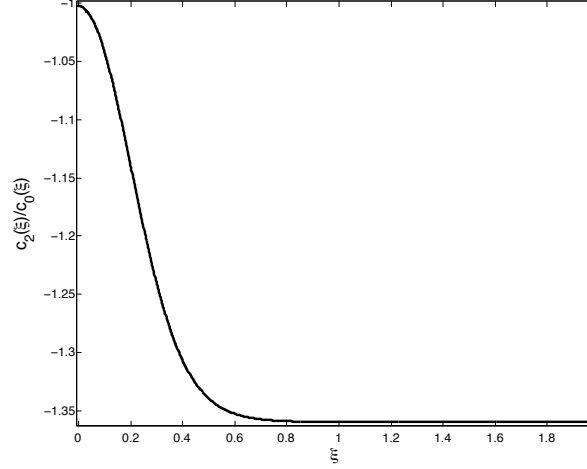


FIG. 6. Ratio between coefficient $c_2(\xi)$ and the coefficient $c_0(\xi)$, as a function of the dimensionless temperature ξ , for the Neumann boundary condition in $d = 4$ dimensions. The plot interpolates between the value $c_2/c_0 = -1.00$ and -1.35 at zero and high temperatures respectively.

B. Non analytic terms: $d = 2$ with $T = 0$, and $d = 3$ with $T > 0$

Let us consider the particular case of $d = 2$ at zero temperature. As shown in Appendix A, for small departures of the plane-plane geometry $\psi(\mathbf{x}_{\parallel}) = a + \eta(\mathbf{x}_{\parallel})$, the correction to the Casimir energy reads, up to second order in η

$$\Gamma_{\infty}^{(2)} = \frac{1}{2} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} [g^{(2)}(k_{\parallel})]_{k_0 \rightarrow 0} |\tilde{\eta}(\mathbf{k}_{\parallel})|^2, \quad (57)$$

with:

$$g^{(2)}(k_{\parallel}) = -2 \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \frac{[p_{\parallel} \cdot (p_{\parallel} + k_{\parallel})]^2}{|p_{\parallel}| |p_{\parallel} + k_{\parallel}|} \frac{1}{1 - e^{-2a|p_{\parallel}|}} \frac{1}{e^{2a|p_{\parallel} + k_{\parallel}|} - 1}. \quad (58)$$

Naively, one would expect the form factor $g^{(2)}(k_{\parallel})$ to admit an expansion in powers of k_{\parallel}^2 , which is the necessary condition in Fourier space to produce a DE in configuration space. However, this is not the case for $d = 2$, as suggested by the fact that the formal expression

$$\frac{\partial g^{(2)}}{\partial k_{\parallel}^2} \Big|_{k_{\parallel} \rightarrow 0} = -2 \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \frac{1}{|p_{\parallel}| (1 - e^{-2a|p_{\parallel}|})} \frac{\partial}{\partial k_{\parallel}^2} \left[\frac{[p_{\parallel} \cdot (p_{\parallel} + k_{\parallel})]^2}{|p_{\parallel}| |p_{\parallel} + k_{\parallel}|} \frac{1}{e^{2a|p_{\parallel} + k_{\parallel}|} - 1} \right]_{k_{\parallel} \rightarrow 0} \quad (59)$$

has an infrared logarithmic divergence at $p_{\parallel} = 0$.

The behavior of $g^{(2)}(k_{\parallel})$ for small values of k_{\parallel} can be determined by studying the integral that defines it in Eq.(58) in the region $a|p_{\parallel}| \ll 1$. In this region, and assuming also that $a|k_{\parallel}| \ll 1$ one can make the approximation

$$\frac{1}{e^{\pm 2x} - 1} \approx \pm \frac{1}{2x} \quad (60)$$

and compute the integrals analytically. In this way, it is possible to show that

$$g^{(2)}(k_{\parallel}) \approx g^{(2)}(0) - \frac{k_{\parallel}^2}{16\pi a^2} \log(k_{\parallel}^2 a^2) + O(k_{\parallel}^2/a^2). \quad (61)$$

This behavior of $g^{(2)}$, that we confirmed with a numerical evaluation of Eq.(58), shows that a *local* DE breaks down for Neumann boundary conditions at $d = 2$. However, one can still perform an expansion for smooth surfaces, including nonlocal contributions in the Casimir energy. For instance, in the present case, the NTLO correction to the PFA will be nonlocal and proportional to

$$\int d^2 x_{\parallel} \eta(x_{\parallel}) \nabla_{\parallel}^2 \log(-a^2 \nabla_{\parallel}^2) \eta(x_{\parallel}). \quad (62)$$

As we will describe more generally in the next Section, the breakdown of the local expansions is related to the existence of massless modes in the theory. These modes are generally allowed by Neumann but not for Dirichlet boundary conditions, that impose a mass gap of order $1/a$.

The logarithmic behavior of the form factor in $d = 2$ induces a similar non-analyticity for $d = 3$ at finite temperature. Indeed, the $m = n = 0$ term in the finite temperature form factor given in Eq.(52) is formally identical to the Neumann form factor $g^{(2)}$ in $d = 2$. Therefore, in an expansion for small values of $|k_{\parallel}|$, in addition to a term proportional to k_{\parallel}^2 , there is a contribution proportional to $(Ta)k_{\parallel}^2 \log(k_{\parallel}^2 a^2)$ at any non-vanishing temperature, which is not cancelled by the rest of the sum over Matsubara frequencies.

V. HIGHER ORDER TERMS IN THE DE

In this Section we discuss some general aspects of the derivative expansion, related with the calculation of higher orders and the eventual breakdown of the expansion.

In this and previous works we considered just the NTLO correction to the PFA, which contains up to two derivatives of the function ψ in the free energy. We expect the next to NTLO (NNTLO) order corrections to contain terms of the form

$$|\partial\psi|^4, \psi|\partial\psi|^2\partial^2\psi, \psi^2\partial^2\psi\partial^2\psi, \psi^2\partial_{\alpha}\partial_{\beta}\psi\partial_{\alpha}\partial_{\beta}\psi, \psi^3\partial^2\partial^2\psi, \quad (63)$$

and terms containing more derivatives for higher orders.

The main question to be answered is whether the free energy can be expanded or not in local terms up to any desired order. For this to hold true, a necessary condition is that the expansion must hold true for a particular case: when $\psi = a + \eta$, with $\eta \ll 1$, and one keeps just the quadratic term in η .

For simplicity, we deal with the $T = 0$ case in $d = 3$ dimensions, an example that will illuminate several aspects of the problem. From the previous sections, we see that the quadratic contribution to the Casimir energy has the form:

$$\Gamma_{\infty}^{(2)} = \frac{1}{2a^5} \int \frac{d^2 \mathbf{k}_{\parallel}}{(2\pi)^2} h^{(2)}(0, a\mathbf{k}_{\parallel}) |\tilde{\eta}(\mathbf{k}_{\parallel})|^2, \quad (64)$$

where the form factor $h^{(2)}$ depends on the boundary conditions of the quantum field. In most of the paper we considered the case of static surfaces, but here it will be useful to discuss the more general case in which the right mirror can be in motion. In this situation, on general grounds, we expect the form factor to be a function of $a(k_0^2 + \mathbf{k}_{\parallel}^2)^{1/2} \equiv a|k_{\parallel}|$, and of course the explicit calculations confirm this fact. Within these approximations, the Casimir energy will not admit an expansion in derivatives if the form factor includes, for instance, odd powers or logs of its argument. Note that at higher orders, and relaxing the condition of a quadratic approximation in η new non-analytic terms may arise, which can be of the same order in the DE as the ones that come from the term quadratic in η .

In 3+1 dimensions, this question can be answered from the explicit expressions of the form factors [13] presented in Appendix B. For Dirichlet boundary conditions, the expansion of the form factor contains, in addition to even powers of the argument, a term proportional to $a^5|k_{\parallel}|^5$ (see Eq.(82)). The non-analytic term becomes a nonlocal contribution in configuration space, that cannot be expanded in derivatives of η . Note that this contribution does not depend on the distance between mirrors. There is a simple interpretation of this term: when considering a flat moving boundary, photon creation produce an imaginary part in the vacuum persistence amplitude when rotated from Euclidean to Minkowski spacetime. Therefore, the expression for $\Gamma_{\infty}^{(2)}$ cannot be analytic in k_{\parallel}^2 . For a single nonrelativistic mirror, this will lead to a dissipative force proportional to the fifth time-derivative of the position. Indeed, the proper analytic continuation of the Euclidean term proportional to $|k_0|^5$ to real time, produces this dissipative force. For the Dirichlet

case, there are no non-analyticities dependent on the distance between mirrors. Physically, this is due to the fact that there is a frequency threshold to produce real photons *between* mirrors, which is of order $1/a$. The conclusion is that, for Dirichlet boundary conditions, the a -dependent part of the DE is well defined up to any order. On the other hand, the a -independent part contains a non-analytic contribution related to the possibility of creating photons of arbitrary low energies from the vacuum.

The situation changes for the case of Neumann boundary condition. As shown in Eq.(84), in addition to the non-analytic term proportional to $|k_{\parallel}|^5 a^5$, which as before produce a contribution to $\Gamma_{\infty}^{(2)}$ that is independent of a , there is a term proportional to $|k_{\parallel}|^3 a^3$. The physical reason of the existence of this term is again clear using the connection with the DCE. Indeed, this term comes from the possibility of creating TM photons between mirrors, moving parallel to the mirrors, for which there is no energy threshold. If the DE is used to compute the force on a moving mirror, this a -dependent dissipative contribution to the force will be missed, i.e. it is only possible to get the dispersive part of the force. On the other hand, for a static and non-flat mirror, this term produces a nonlocal component of the force, which will be smaller than the NTLO, but the dominant part of the NNTLO.

It is interesting to remark that both the non-analytic contributions proportional to $|k_{\parallel}|^5$ for Dirichlet and Neumann boundary conditions in $3 + 1$ dimensions, and the $|k_{\parallel}|^3$ term for the Neumann case, could be derived from the form factors described in the previous section by analyzing the infrared behavior of the integrals in p_{\parallel} , as we did in Sec. IV B. Moreover, it is clear that all of them have the same physical origin: the existence of massless degrees of freedom. For Neumann boundary conditions at $T = 0$, the non-analyticities show up in the NTLO for $d = 2$, and in the NNTLO for $d = 3$. For Dirichlet boundary conditions the non-analytic term is independent of the distance between mirrors, and therefore does not contribute to the Casimir force.

The situation is analogous to that of effective field theories that involve massless particles. In that case, in addition to local terms in the effective action, there are nonlocal (or non-analytic contributions) which can be interpreted as arising from the fact that there is no threshold for creating such particles [9]. A prototypical example is quantum field theory under the influence of external (classical) backgrounds. For massive quantum fields in curved spaces [14], the effective action and the energy momentum tensor of the quantum fields can be approximated by a DE (usually known as the Schwinger DeWitt expansion in that context). Each subsequent term in the expansion contains additional derivatives of the metric and inverse powers of the mass of the quantum field. The expansion is valid as long as the typical scale of variation of the classical background is much larger than the inverse mass. However, for massless quantum fields, it is necessary to consider nonlocal contributions. In our case, the role of the background is played by the curved surface, and there are both massive and massless excitations: the massive ones are the Dirichlet modes (TE photons) inside the "cavity". The massless ones are the Neumann modes (TM photons) with momentum in the direction parallel to the plates, and TE and TM photons outside the cavity.

The physical picture suggests possible ways out to improve the PFA even beyond the NTLO correction. This would involve a separate treatment of massless and massive excitations. We hope to address this issue in a future work.

VI. CONCLUSIONS

We have obtained explicit expressions for the NTLO term in a DE for the Casimir free energy for a real scalar field in d spatial dimensions. The field satisfies either Dirichlet or Neumann boundary conditions on two static mirrors, one of them flat and located at the $x_d = 0$ plane, while the other is described by the equation: $x_d = \psi(x_1, x_2, \dots, x_{d-1})$. We have shown that, for Dirichlet boundary conditions, the NTLO term in the Casimir energy is always of quadratic order in derivatives, regardless of the number of dimensions. Therefore it is local, and determined by a single coefficient. We evaluated the ratio between that coefficient and the one for the PFA term, for different values of d at zero and high temperatures.

We have also shown that the same holds true, if $d \neq 2$, for a field which satisfies Neumann conditions. When $d = 2$, the NTLO term becomes nonlocal in coordinate space, which is a clear manifestation of the existence of gapless excitations allowed by the Neumann conditions [15]. It may be seen that among all the possible combination of linear boundary conditions on the mirrors, just this case, Neumann conditions on both mirrors, can produce these modes.

When including thermal effects, we have shown that, for Dirichlet mirrors, the NTLO term in the free energy is also well defined (local) for any temperature T . Besides, it interpolates between the proper limits: namely, when $T \rightarrow 0$ it tends to the one we had calculated for the Casimir energy, while for $T \rightarrow \infty$ it corresponds to the one for a $d = 2$ theory, realizing the expected dimensional reduction at high temperatures. On the contrary, for Neumann mirrors in $d = 3$, we found a nonlocal NTLO term for any $T > 0$, which vanishes linearly when $T \rightarrow 0$. This leaves room, when the temperature is sufficiently low, to use just the local term (of second order in derivatives) as the main correction to the PFA. But of course, the nonlocal term will always break down for a higher temperature, whose value will depend on the actual shape of the surface involved. We stress once more that this non-analytic behavior is a consequence of

the Neumann boundary conditions, and may not be present for imperfect boundary conditions, as those considered in Ref.[6].

In the course of our derivations we have obtained integral expressions for the momentum space kernels which determine the quadratic contribution to the free energy for small departure from the planar case. Those kernels are well defined in any number of spatial dimensions and temperatures, and agree with the known results for $d = 3$ and $T = 0$ [13]. They can be used to extract the NTLO terms, be they local or nonlocal. Although for the static cases we have considered in this article they are only needed for time independent configurations, we also present the expressions for the kernels at non-zero frequencies.

APPENDIX A: EXPANSION TO ORDER $\mathcal{O}(\eta^2)$, NEUMANN CASE.

We present here the main steps and intermediate results corresponding to the calculation of the free energy up to the second order in the function η , which measures the departure from the planar case. We assume that $\psi(x_{\parallel}) = a + \eta(x_{\parallel})$, with a equaling the average of ψ . Namely, we want to construct the terms in

$$\Gamma_{\beta} = \Gamma_{\beta}^{(0)} + \Gamma_{\beta}^{(1)} + \Gamma_{\beta}^{(2)} + \dots \quad (65)$$

where the index denotes the order in η . The term of order 1 vanishes, and, in terms of the expanded matrix elements of \mathbb{U} , we may write the more explicit expressions:

$$\begin{aligned} \Gamma_{\beta}^{(0)} &= \frac{1}{2\beta} \text{Tr}[\log \mathbb{U}^{(0)}] \\ \Gamma_{\beta}^{(2)} &= \Gamma_{\beta}^{(2,1)} + \Gamma_{\beta}^{(2,2)}, \end{aligned} \quad (66)$$

where:

$$\begin{aligned} \Gamma_{\beta}^{(2,1)} &= \frac{1}{2\beta} \text{Tr}[(\mathbb{U}^{(0)})^{-1} \mathbb{U}^{(2)}] \\ \Gamma_{\beta}^{(2,2)} &= -\frac{1}{4\beta} \text{Tr}[(\mathbb{U}^{(0)})^{-1} \mathbb{U}^{(1)} (\mathbb{U}^{(0)})^{-1} \mathbb{U}^{(1)}]. \end{aligned} \quad (67)$$

In order to simplify, and at the same time render the expressions more compact, we shall keep the 0 component of momenta to appear below continuous, as if they corresponded to zero temperature. In order to obtain the proper finite temperature expressions one should just replace integrals over the 0 component of the momenta by sums over Matsubara frequencies. Namely,

$$\int \frac{dk_0}{2\pi} \dots A(k_0, \dots) \rightarrow \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} A(\omega_n, \dots). \quad (68)$$

Let us consider the explicit form of $(\mathbb{U}^{(0)})$ and its inverse, since both of them are required to construct the terms contributing to Γ_{β} above. We first note that the zero order term is given by:

$$\mathbb{U}^{(0)}(x_{\parallel}, x'_{\parallel}) = \begin{pmatrix} \partial_d \partial'_d \Delta(x - x')|_{x_d, x'_d \rightarrow 0} & \partial_d \partial'_d \Delta(x - x')|_{x_d \rightarrow 0, x'_d \rightarrow a} \\ \partial_d \partial'_d \Delta(x - x')|_{x_d \rightarrow a, x'_d \rightarrow 0} & \partial_d \partial'_d \Delta(x - x')|_{x_d \rightarrow a, x'_d \rightarrow a} \end{pmatrix}, \quad (69)$$

which, because of its independence of η , can be conveniently Fourier transformed in the parallel coordinates:

$$\mathbb{U}^{(0)}(x_{\parallel}, x'_{\parallel}) = \mathbb{U}^{(0)}(x_{\parallel} - x'_{\parallel}) = \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{ik_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} \tilde{\mathbb{U}}^{(0)}(k_{\parallel}), \quad (70)$$

where

$$\tilde{\mathbb{U}}^{(0)}(k_{\parallel}) = -\frac{|k_{\parallel}|}{2} \begin{pmatrix} 1 & e^{-|k_{\parallel}|a} \\ e^{-|k_{\parallel}|a} & 1 \end{pmatrix}. \quad (71)$$

Thus:

$$(\mathbb{U}^{(0)})^{-1}(x_{\parallel} - x'_{\parallel}) = \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{ik_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} \frac{(-2)}{|k_{\parallel}|(1 - e^{-2|k_{\parallel}|a})} \begin{pmatrix} 1 & -e^{-|k_{\parallel}|a} \\ -e^{-|k_{\parallel}|a} & 1 \end{pmatrix}. \quad (72)$$

Regarding $\mathbb{U}^{(1)}$, we see that

$$\mathbb{U}_{RR}^{(1)} = \mathbb{U}_{LL}^{(1)} = 0, \quad (73)$$

and

$$\begin{aligned} \mathbb{U}_{LR}^{(1)}(x_{\parallel}, x'_{\parallel}) &= \frac{1}{2} \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{ik_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} [|k_{\parallel}|^2 \eta(x_{\parallel}) + ik^{\alpha} \partial_{\alpha} \eta(x_{\parallel})] e^{-|k_{\parallel}|a} \\ &= \mathbb{U}_{RL}^{(1)}(x'_{\parallel}, x_{\parallel}). \end{aligned} \quad (74)$$

Finally, to the second order, $\mathbb{U}_{LL}^{(2)} = 0$, and:

$$\mathbb{U}_{RR}^{(2)}(x_{\parallel}, x'_{\parallel}) = \frac{1}{2} \int \frac{d^d k_{\parallel}}{(2\pi)^d} e^{ik_{\parallel} \cdot (x_{\parallel} - x'_{\parallel})} (|k_{\parallel}|^3 + \frac{k^{\alpha} k^{\beta}}{|k_{\parallel}|} \partial_{\alpha} \partial_{\beta}) \eta(x_{\parallel}) \eta(x'_{\parallel}). \quad (75)$$

Regarding $\mathbb{U}_{LR}^{(2)}$ and $\mathbb{U}_{RL}^{(2)}$, they are non-vanishing, but it may be seen that they do not contribute to the second order term.

Thus,

$$\begin{aligned} \Gamma_{\beta}^{(2,1)} &= \frac{1}{2\beta} \text{Tr} \left[(\mathbb{U}_{RR}^{(0)})^{-1} \mathbb{U}_{RR}^{(2)} \right] \\ &= \frac{1}{2\beta} \int d^d x_{\parallel} d^d x'_{\parallel} (\mathbb{U}_{RR}^{(0)})^{-1}(x_{\parallel}, x'_{\parallel}) \mathbb{U}_{RR}^{(2)}(x'_{\parallel}, x_{\parallel}), \end{aligned} \quad (76)$$

and (using the properties of the matrix elements under the exchange of arguments):

$$\begin{aligned} \Gamma_{\beta}^{(2,2)} &= -\frac{1}{2\beta} \text{Tr} \left[(\mathbb{U}_{LL}^{(0)})^{-1} \mathbb{U}_{LR}^{(1)} (\mathbb{U}_{RR}^{(0)})^{-1} \mathbb{U}_{RL}^{(1)} \right] \\ &\quad - \frac{1}{2\beta} \text{Tr} \left[(\mathbb{U}_{RL}^{(0)})^{-1} \mathbb{U}_{LR}^{(1)} (\mathbb{U}_{RL}^{(0)})^{-1} \mathbb{U}_{LR}^{(1)} \right]. \end{aligned} \quad (77)$$

In Fourier space, after some algebra, one then finds:

$$\Gamma_{\beta}^{(2)} = \frac{1}{2} \int \frac{d^d k_{\parallel}}{(2\pi)^d} g^{(2)}(k_{\parallel}) |\eta(k_{\parallel})|^2, \quad (78)$$

with:

$$g^{(2)}(k_{\parallel}) = -2 \int \frac{d^d p_{\parallel}}{(2\pi)^d} \frac{[p_{\parallel} \cdot (p_{\parallel} + k_{\parallel})]^2}{|p_{\parallel}| |p_{\parallel} + k_{\parallel}|} \frac{1}{1 - e^{-2a|p_{\parallel}|}} \frac{1}{e^{2a|p_{\parallel} + k_{\parallel}|} - 1}. \quad (79)$$

APPENDIX B: EXACT EXPRESSIONS FOR THE FORM FACTORS IN 3 + 1 DIMENSIONS.

In this Appendix we present exact expressions and series expansions for the form factors $f^{(2)}(k_{\parallel})$ and $g^{(2)}(k_{\parallel})$ at zero temperature and $d = 3$. These expressions have been previously obtained in Ref.[13] (see also Ref.[16]). We will use the dimensionless quantity $x = a|k_{\parallel}|$.

For Dirichlet boundary conditions the form factor $f^{(2)}$ reads

$$f^{(2)}(k_{\parallel}) = -2 \int \frac{d^3 p_{\parallel}}{(2\pi)^3} \frac{|p_{\parallel}| |p_{\parallel} + k_{\parallel}|}{(1 - e^{-2a|p_{\parallel}|})(e^{2a|p_{\parallel} + k_{\parallel}|} - 1)}. \quad (80)$$

An explicit evaluation gives [13]:

$$\begin{aligned} a^5 f^{(2)}(x) &= -\frac{x^3 \text{Li}_2(e^{-2x})}{48\pi^2} - \frac{x^2 \text{Li}_3(e^{-2x})}{24\pi^2} - \frac{x \text{Li}_4(e^{-2x})}{16\pi^2} - \frac{\text{Li}_5(e^{-2x})}{16\pi^2} - \frac{\pi^2 \text{Li}_2(1 - e^{-2x})}{240x} \\ &\quad + \frac{\frac{\pi^6}{945} - \text{Li}_6(e^{-2x})}{32\pi^2 x} + \frac{x^4 \log(1 - e^{-2x})}{120\pi^2} - \frac{\pi^2 x}{240}, \end{aligned} \quad (81)$$

where Li_n denote Polylogarithm functions. This result can be expanded in powers of x as follows:

$$a^5 f^{(2)}(x) \approx -\frac{\pi^2}{120} - \frac{\pi^2 x^2}{1080} + \frac{(45 + \pi^4) x^4}{27000\pi^2} - \frac{x^5}{720\pi^2} + \frac{(315 - 2\pi^4) x^6}{793800\pi^2} + \frac{(\pi^4 - 105) x^8}{5103000\pi^2} + \frac{\left(\frac{8}{165} - \frac{8\pi^4}{16335}\right) x^{10}}{30240\pi^2} + O(x^{12}) \quad (82)$$

which shows the presence of a non-analytic term proportional to x^5 , which is independent of a .

For Neumann boundary conditions the form factor $g^{(2)}$ is given in Eq.(79) with $d = 3$. Evaluating explicitly this integral it is possible to show that [13]

$$a^5 g^{(2)}(x) = \frac{1}{24} \left(\frac{x^2}{2\pi^2} + 1 \right) x \text{Li}_2(e^{-2x}) + \left(\frac{1}{16} - \frac{x^2}{48\pi^2} \right) \text{Li}_3(e^{-2x}) - \frac{5x \text{Li}_4(e^{-2x})}{32\pi^2} - \frac{7\text{Li}_5(e^{-2x})}{32\pi^2} - \frac{\pi^2 \text{Li}_2(1 - e^{-2x})}{240x} + \frac{\pi^2 \text{Li}_4(e^{-2x})}{32\pi^2 x} - \frac{7\text{Li}_6(e^{-2x})}{2} - \frac{\pi^6}{135} + \frac{x^4 \log(1 - e^{-2x})}{120\pi^2} - \frac{\pi^2 x}{720}. \quad (83)$$

Expanding this result for $x \ll 1$ we obtain

$$a^5 g^{(2)}(x) \approx -\frac{\pi^2}{120} + \frac{(30 - \pi^2) x^2}{1080} - \frac{x^3}{64} + \frac{(1095 + 50\pi^2 + \pi^4) x^4}{27000\pi^2} - \frac{11x^5}{720\pi^2} + \frac{(2205 - 42\pi^2 - 2\pi^4) x^6}{793800\pi^2} + \frac{(100\pi^2 + 7\pi^4 - 3045) x^8}{35721000\pi^2} + 4 \left(\frac{47}{38102400\pi^2} - \frac{1}{23328000} - \frac{\pi^2}{246985200} - \frac{\frac{1}{1403325} + \frac{\pi^2}{4677750}}{128\pi^2} \right) x^{10} + O(x^{12}). \quad (84)$$

We see that the Neumann form factor has non-analytic terms proportional to x^3 and x^5 .

ACKNOWLEDGEMENTS

This work was supported by ANPCyT, CONICET, UBA and UNCuyo.

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